GAIN MARGIN, PHASE MARGIN, and TRANSIENT RESPONSE

of SOME SIMPLE SERVOMECHANISMS

by

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This paper will consider some simple servomechanism transfer functions and show relations between Gain and Phase margins for systems using these transfer functions. Transient response to a step input will be considered as a function of phase margin.

Transient response was determined by means of the Moore School Differential Analyzer. Due to availability of the differential analyzer this analysis is concerned with a non-linear servomechanism system in that the effective error signal cannot exceed one in value. The applied error signal ε_c will be 5, the saturated value of the error signal ε_c will equal one. This case is analogous to that in which the servo amplifier saturates at $\varepsilon_c = 1$ while $\theta_i - \theta_o = 5$.

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Transient response was determined by means of the Moore School Differential Analyzer. Due to availability of the differential analyzer this analysis is concerned with a non-linear servomechanism system in that the effective error signal cannot exceed one in value. The applied error signal $\mathcal{E}_c$ will be 5, the saturated value of the error signal $\mathcal{E}_c$ will equal one. This case is analogous to that in which the servo amplifier saturates at $\mathcal{E}_c = 1$ while $\theta_i - \theta_o = 5$.

A simple feedback system of the form shown may by considered to consist of an input θ_i, an error measuring device A, a transfer function KY(p) where $p = j\omega$, an output θ_o and a feedback line. Let $\theta_i - \theta_o = \mathcal{E}$. Then $\theta_o = \mathcal{E}\,(KY(p)) = [KY(p)][\theta_i - \theta_o]$ or $\theta_o/\theta_i = \dfrac{KY(p)}{1 + KY(p)}$. It has been shown (ref. c) that a transfer function may be reduced to the form $\dfrac{A_0 + A_1 p + A_2 p^2 + \ldots\ldots A_m p^m}{B_0 + B_1 p + B_2 p^2 + \ldots\ldots B_m p^m}$

The transfer functions considered herein will be of the form:

$$\text{(A)} \qquad KY(p) = \frac{K}{p(p+1)}$$

$$\text{(B)} \qquad KY(p) = \frac{K}{p(p+1)(\alpha p+1)(}$$

$$\text{(C)} \qquad KY(p) = \frac{K}{p(p+1)(\alpha p+1)(\beta p+1)}$$

These transfer functions might arise as follows:

(A) Components (a) synchro transformer $E(j\omega)/\mathcal{E}(j\omega) = K_1$

(b) D.C. motor with shunt field control

$$\theta_o/I_f = \frac{K_2}{(j\omega)(j\omega\tau+1)}$$

(c) electronic amplifier

Here $\theta_o/\mathcal{E} = \dfrac{K_1K_2}{j\omega(j\omega\tau+1)}$

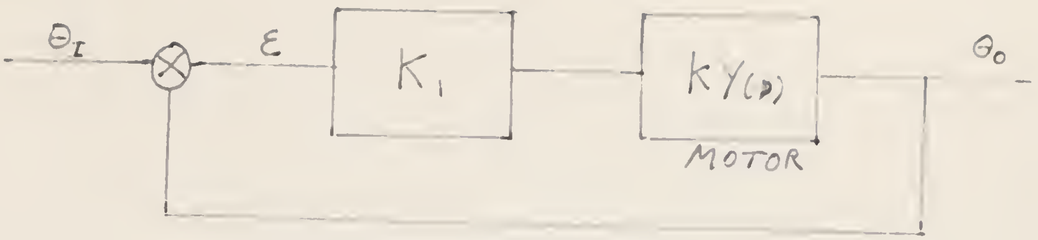
[illegible] to the dimensionless form of Eq.(A), let $j\omega\tau = p$, then

$$\theta_o/\mathcal{E} = \frac{K_1K_2\tau}{p(p+1)} = \frac{K}{p(p+1)} = KY(p) \qquad p = j\omega$$

[illegible] K is a [illegible], hereinafter called the gain constant.

Transfer functions (B) and (C) may be similarly derived, as for example

(B)



$$KY(jw) = \frac{K_1 K_2 K_f / R_f f}{jw(\tau_f\, jw+1)(\tau_m\, jw+1)} = \frac{K_1 K_2 K_f / R_f f}{jw(jw\tau_1 + 1)(jw\tau_2 + 1)}$$

arranging τ's in order of size largest τ being τ_1.

Let $w\tau_1 = u$, then $KY(ju) = \dfrac{K}{ju(ju+1)(ju\alpha+1)}$

where $\alpha = \dfrac{\tau_2}{\tau_1}$. Note that $\alpha \leq 1$.

Expression (C) may be similarly obtained. Thus, in the following discussion α and β will have values $0 \leq \alpha \leq 1$; $0 \leq \beta \leq 1$.

The transient response of systems having transfer functions of these forms may be obtained by means of the La Place transform. Thus for expression (A):

$$\theta_o/\theta_i = \frac{KY(p)}{1+KY(p)} = \frac{K}{p(p+1) + K}$$

For θ_i a step function and $K = 1$,

$$\theta_o(p) = \frac{1}{p(p^2+p+1)} = \frac{1}{p\left[(p+\frac{1}{2})^2 + \frac{3}{4}\right]}$$

whence (form 1.304), ref(d).

$$\theta_o(t) = 1 + 2/\sqrt{3}\, e^{-\frac{1}{2}t} \sin(\sqrt{3}/2\ t - \psi) \text{ where } \psi = \tan^{-1}\sqrt{3}.$$

Where a large number of transient responses are required it is convenient to use the differential analyzer. This problem may be set up using the operator $p = d\theta/dt$. For the transfer function

(A) $KY(p) = \dfrac{K}{p(p+1)} = \theta_o/\mathcal{E}$.

$p(p+1)\theta_o = K\mathcal{E}$ $\quad (d^2/dt^2 + d/dt)\,\theta_o = K\mathcal{E}$

$\mathcal{E} = \theta_i - \theta_o$ $\quad$ For θ_i a constant $(d^2/dt^2 + d/dt)\theta_o = -\left(\frac{d^2}{dt^2} + \frac{d}{dt}\right)\mathcal{E}$

then $-\left(\frac{d^2}{dt^2}+\frac{d}{dt}\right)\mathcal{E}(t) = K\,\mathcal{E}(t)$ The solution to this equation gives $\mathcal{E}(t)$ whence $\theta_o(t) = 1 - \mathcal{E}(t)$

For expression (B) $KY(p) = \dfrac{K}{p(p+1)(\alpha p+1)} = \theta_o/\mathcal{E}$

$$[\alpha p^3 + (1+\alpha)p^2 + p]\,\theta_o(t) = K\,\mathcal{E}(t)$$

$$-[\alpha p^3 + (1+\alpha)p^2 + p]\,\mathcal{E}(t) = K\,\mathcal{E}(t)$$

the solution of which gives $\mathcal{E}(t)$, whence $\theta_o(t) = 1 - \mathcal{E}(t)$

For expression (C) $KY(p) = \dfrac{K}{p(p+1)(\alpha p+1)(\beta p+1)}$

$$\theta_o/\mathcal{E} = \frac{K}{\alpha\beta p^4 + (\alpha\beta+\alpha+\beta)p^3 + (\alpha+\beta+1)p^2 + p}$$

$$-[\alpha\beta p^4 + (\alpha\beta+\alpha+\beta)p^3 + (\alpha+\beta+1)p^2 + p]\,\mathcal{E}(t) = K\,\mathcal{E}(t)$$

whence $\theta_o(t) = 1 - \mathcal{E}(t)$

As transient response to a step input will be discussedin terms of gain margin and phase margin these will be defined as:

(a) Gain Margin: $1 - |KY(p)|$ at the point where the angle of $KY(p)$ equals 180°.

(b) Phase Margin: angle of $KY(p)$ diminished by 180° at the point where $|KY(p)| = 1$.

The servomechanisms considered will have the transfer functions:

(A) $KY(p) = \dfrac{K}{p(p+1)}$

(B) $KY(p) = \dfrac{K}{p(p+1)(\alpha p+1)}$

(C) $KY(p) = \dfrac{K}{p(p+1)(\alpha p+1)(\beta p+1)}$

Their transient responses to a step input will be discussed as functions of phase margin and the relations between gain and phase margins for various values of α, β, and K will be shown. As previously stated, these servomechanisms will be considered to be non-linear in that the effective error signal cannot exceed one. The applied error signal $\mathcal{E}_o$ will be 5, the saturated value of the error signal $\mathcal{E}_c$ will equal one. The effects of saturation on transient response are shown in curve A-1.

To show the relations between gain and phase margins and the transient response to a step input it is necessary to determine the K required for each value of gain and phase margin considered. This is done by solving for the frequency u required to obtain the proper angle of KY(p), then substituting this value in KY(p) to determine K.

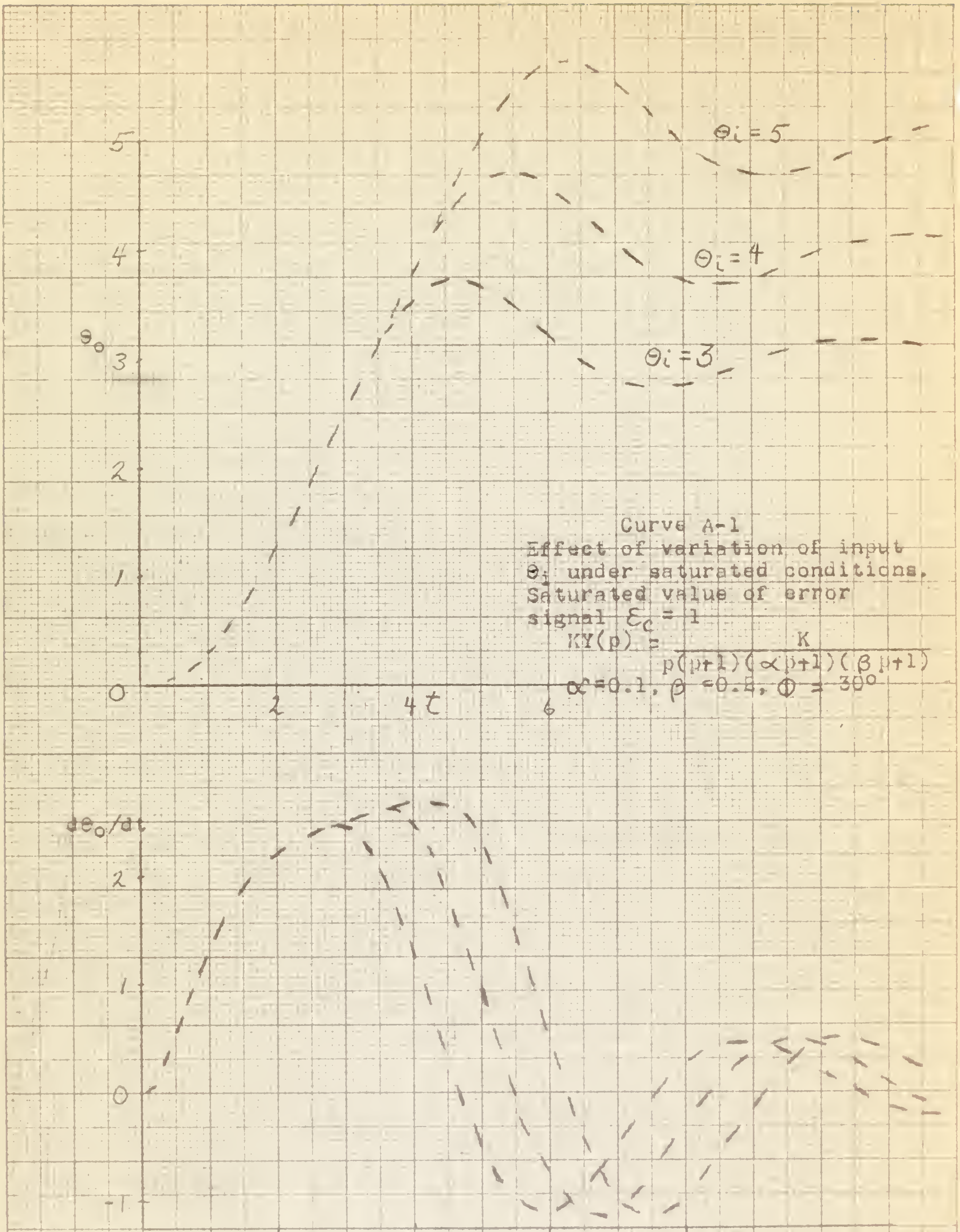
For expression (A) it is seen that gain margin does not exist since the locus of KY(p) does not cross the 180^0 axis. At the point at which phase margin is measured,

$$\left| KY(p) \right| = \left| \frac{K}{p(p+1)} \right| = \left| \frac{K}{ju(ju+1)} \right| = 1$$

$$K = \left| u\sqrt{u^2+1} \right| \qquad \text{angle } KY(p) = -90 - \tan^{-1}u$$

$$\begin{aligned} \text{phase margin } \emptyset &= \angle KY(p) - 180 \\ &= -90 - \tan^{-1}u - 180 \\ &= -270 - \tan^{-1} u \text{ or } \tan(\emptyset + 270) = -u_o \end{aligned}$$

$\cot \emptyset = u_o$ then $K = u_o\sqrt{u_o^2 +1}$



Curve A-1
Effect of variation of input θ_i under saturated conditions.
Saturated value of error signal $\varepsilon_c = 1$

$$KY(p) = \frac{K}{p(p+1)(\alpha p+1)(\beta p+1)}$$

$\alpha = 0.1,\ \beta = 0.2,\ \Phi = 30^\circ$

Expression (B)

Phase Margin: $KY(p) = \dfrac{K}{p(p+1)(\alpha p+1)} = \dfrac{K}{ju(ju+1)(j\alpha u+1)}$

$\angle KY(p) = \theta = -90 - \tan^{-1}u - \tan^{-1}\alpha u$

phase margin $\Phi = -90 - \tan^{-1}\dfrac{u(1+\alpha)}{1-\alpha u^2} - 180$

$\tan(\Phi + 270) = -\cot\Phi = \dfrac{-u(1+\alpha)}{1-\alpha u^2}$

$\cot\Phi - \alpha u^2\cot\Phi - u(1+\alpha) = 0$

$u^2 + u\dfrac{(1+\alpha)}{\alpha}\tan\Phi - 1/\alpha = 0$

Substitution of the various phase margins Φ and α's considered and solution for u gives u_o for the desired phase margin. Then: $\left|KY(ju_o)\right| = 1 = \dfrac{K}{u_o(u_o^2+1)^{\frac{1}{2}}(\alpha^2u_o^2+1)^{\frac{1}{2}}}$

$= \dfrac{K\cos\theta_1\cos\theta_2}{u_o}$

or $K = \dfrac{u_o}{\cos\theta_1\cos\theta_2}$ where $\theta_1 = \tan^{-1}u_o$
$\theta_2 = \tan^{-1}\alpha u_o$

gives the value of K for the desired phase margin Φ.

Gain Margin:

gain margin $= 1 - \left|KY(p)\right|$ where $\angle KY(p) = 180^o$

$\angle KY(p) = -90 - \tan^{-1}u - \tan^{-1}\alpha u = 180$

$\tan^{-1}u + \tan^{-1}\alpha u = -270$ $\qquad \tan\dfrac{u(1+\alpha)}{1-\alpha u^2} = -270$

$\dfrac{u(1+\alpha)}{1-\alpha u^2} = \tan(-270) = \tan 90 = \infty$

therefore $1-\alpha u^2 = 0$ or $\alpha u^2 = 1$ whence $u_1 = \dfrac{1}{\sqrt{\alpha}}$

then $\left|KY(p)\right| = \dfrac{K}{u_1\sqrt{u_1^2+1}\sqrt{\alpha^2u_1^2+1}} = \dfrac{K\cos\theta_1\cos\theta_2}{u_1}$

where $\theta_1 = \tan^{-1}u_1$; $\theta_2 = \tan^{-1}\alpha u_1$

$1 - \left|KY(ju_1)\right| =$ gain margin

Expression (C)

Phase Margin: $KY(ju) = \dfrac{K}{ju(ju+1)(j\alpha u+1)(j\beta u+1)}$

$\angle KY(ju) = \theta = -90 - \tan^{-1}u - \tan^{-1}\alpha u - \tan^{-1}\beta u$

$$\theta + 90 = -\tan^{-1}u - \tan^{-1}\frac{u(\alpha+\beta)}{1-\alpha\beta u^2}$$

$$= -\tan^{-1}\frac{u + \dfrac{u(\alpha+\beta)}{1-\alpha\beta u^2}}{1 - \dfrac{u^2(\alpha+\beta)}{1-\alpha\beta u^2}} \qquad \Phi = \theta - 180^\circ$$

$\cot\Phi = \cot\theta = \dfrac{u(1-\alpha\beta u^2) + u(\alpha+\beta)}{1-\alpha\beta u^2 - u^2(\alpha+\beta)}$ which reduces to

$$u^3 - \frac{u^2(\alpha\beta+\alpha+\beta)\cot\Phi}{\alpha\beta} - \frac{u(1+\alpha+\beta)}{\alpha\beta} + \frac{\cot\Phi}{\alpha\beta} = 0$$

Solution for u for given values of Φ, α, and β gives u_o

Then, since $|KY(ju)| = 1$ at $u = u_o$

$K = \dfrac{u_o}{\cos\theta_1 \cos\theta_2 \cos\theta_3}$ where $\theta_1 = \tan^{-1} u_o$

$\theta_2 = \tan^{-1}\alpha u_o$

$\theta_3 = \tan^{-1}\beta u_o$

Gain Margin: Previously, $\cot\Phi = \dfrac{u(1-\alpha\beta u^2) + u(\alpha+\beta)}{1- \quad u^2 - u^2(\quad - \quad)}$

$$= \frac{u(1-\alpha\beta u^2) + u(\alpha+\beta)}{1-u^2(\alpha\beta+\alpha+\beta)}$$

At $\angle KY(p) = 180$ $\cot\Phi = \infty$, therefore $u^2 = \dfrac{1}{\alpha\beta+\alpha+\beta}$

or $u_1 = \dfrac{1}{\sqrt{\alpha\beta+\alpha+\beta}}$. Then, as for (B)

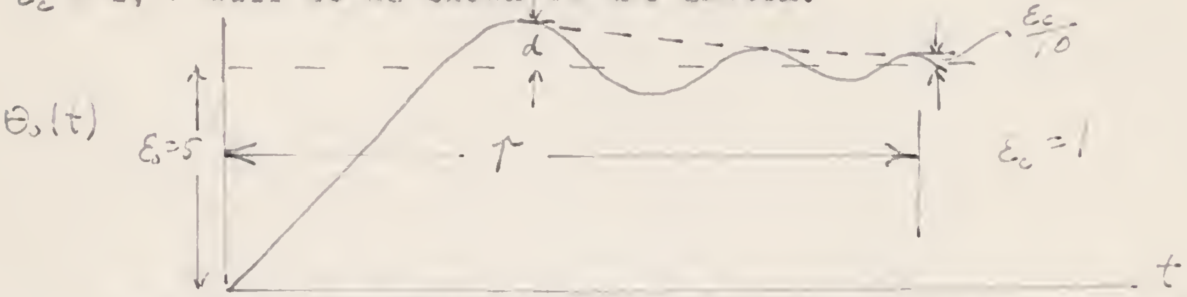
$1-(\text{gain margin}) = |KY(p)| = \dfrac{K\cos\theta_1\cos\theta_2\cos\theta_3}{u_1}$

where $\theta_1 = \tan^{-1} u_1$
$\theta_2 = \tan^{-1}\alpha u_1$
$\theta_3 = \tan^{-1}\beta u_1$

For expressions (B) and (C) values of α of 0.1 through 1.0, and β of 0.1 through 1.0 and phase margins of 15°, 30°, 45°, 60°, and 75° are considered. Curves of constant phase margin as a function of α vs. K are given for (B) and of β vs. K for values of $\alpha = .1, .4$, and .8 for (C). (Curves B-1, C-1 through C-4) Superimposed thereupon are curves of constant gain margins. These curves show the relation between gain and phase margins for the transfer functions considered. The low values of K associated with an uncompensated system are evident.

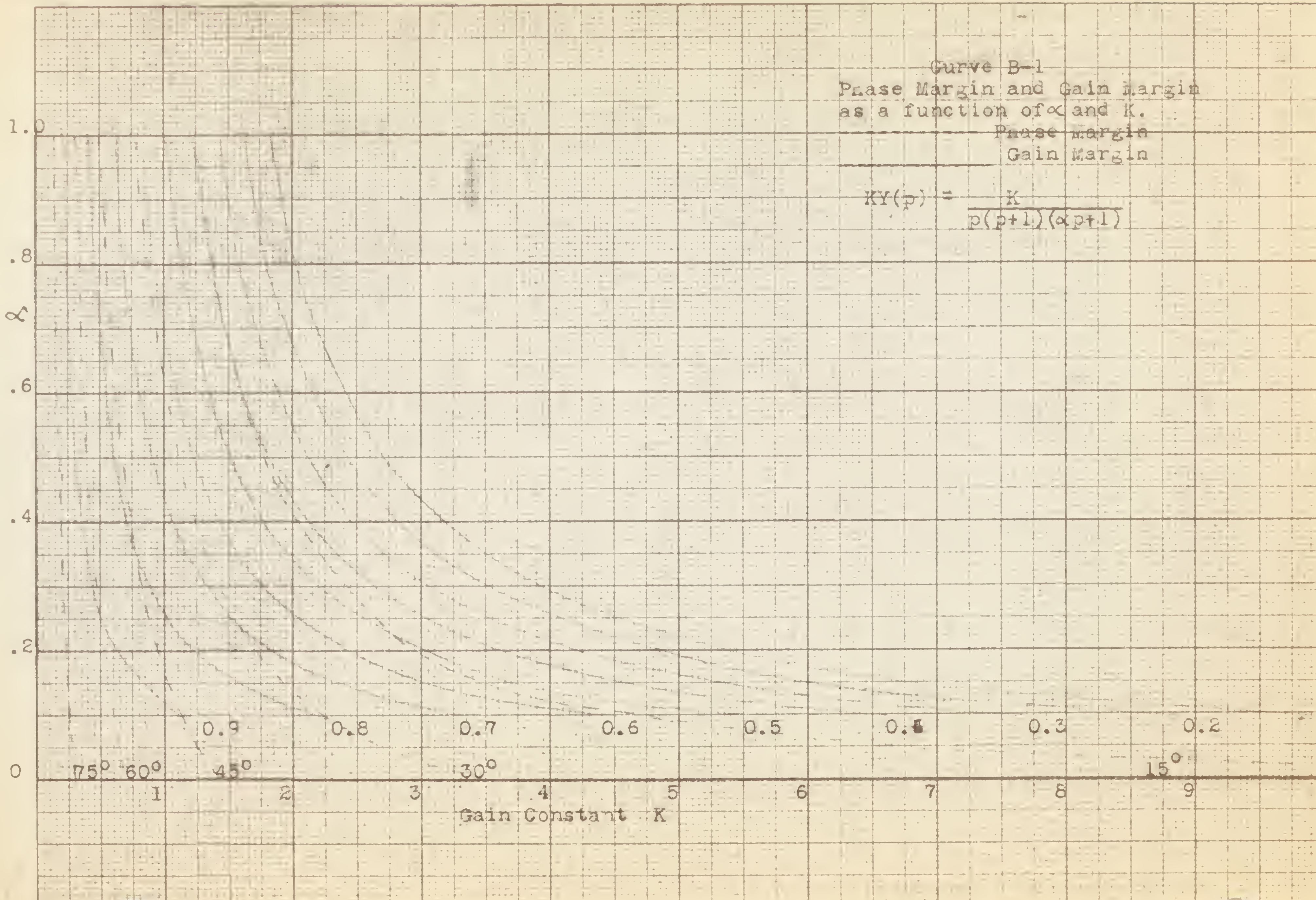
TRANSIENT RESPONSE

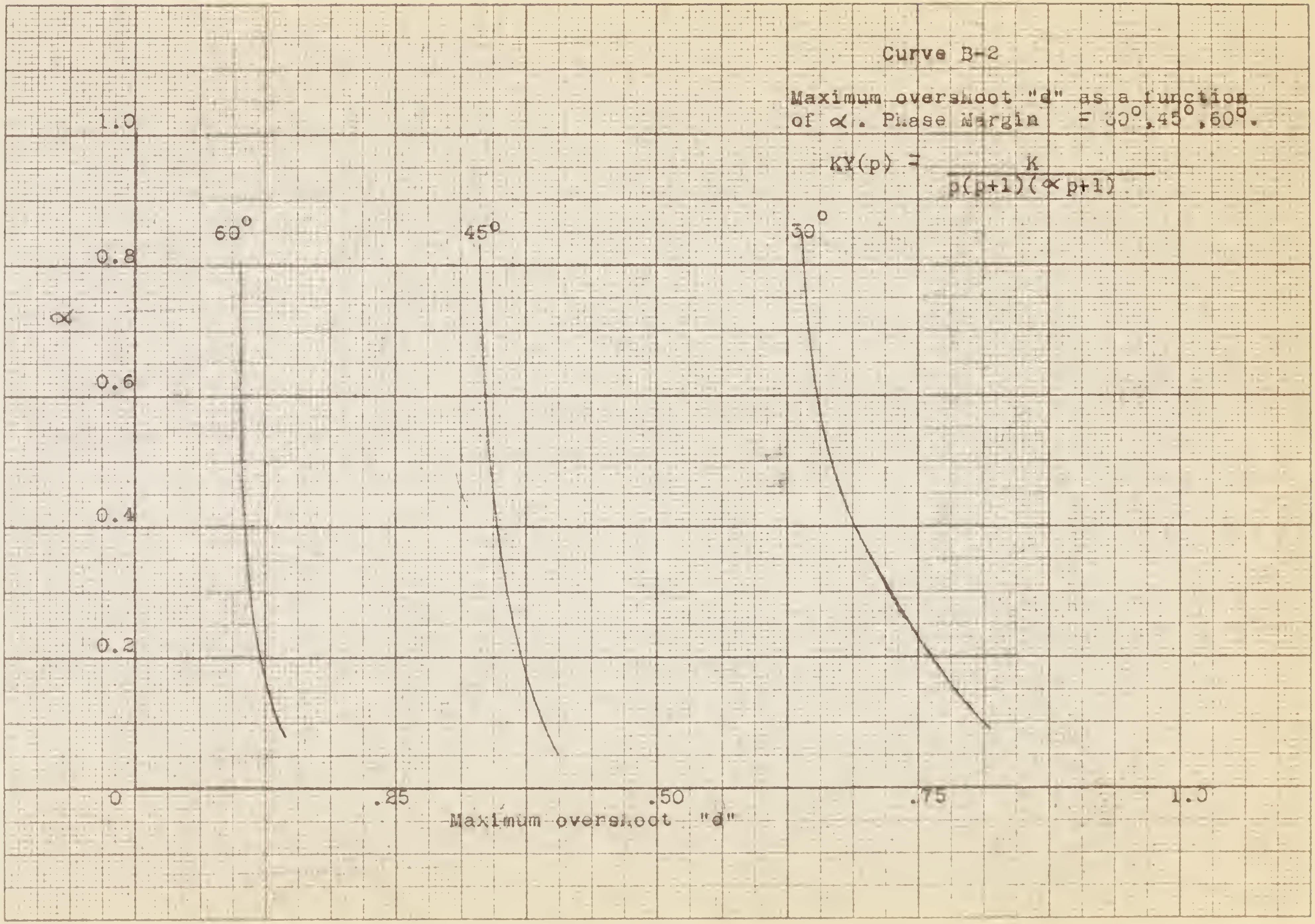
Of principal interest in the transient response to a step input of most servomechanisms are the amount of overshoot "d" and the time τ required for the output of the system to arrive at or sufficiently near to the value of the input. Herein, time τ will be the time required for the output of the system to arrive at one-tenth of the saturation value of the error signal. Since $\mathcal{E}_c = 1$, τ will be as shown in the sketch.



Curve B-2 gives the value of maximum overshoot "d" for the servo of expression (B) as a function of α. "d" is seen to vary more significantly with phase margin than with α. For phase margins of 45° and 60° "d" is essentially independent of α between $\alpha = .1$ and $\alpha = 1.0$

Curve B-3 shows τ, time to reach 1/10 saturation value of error signal as a function of α for various phase margins for the

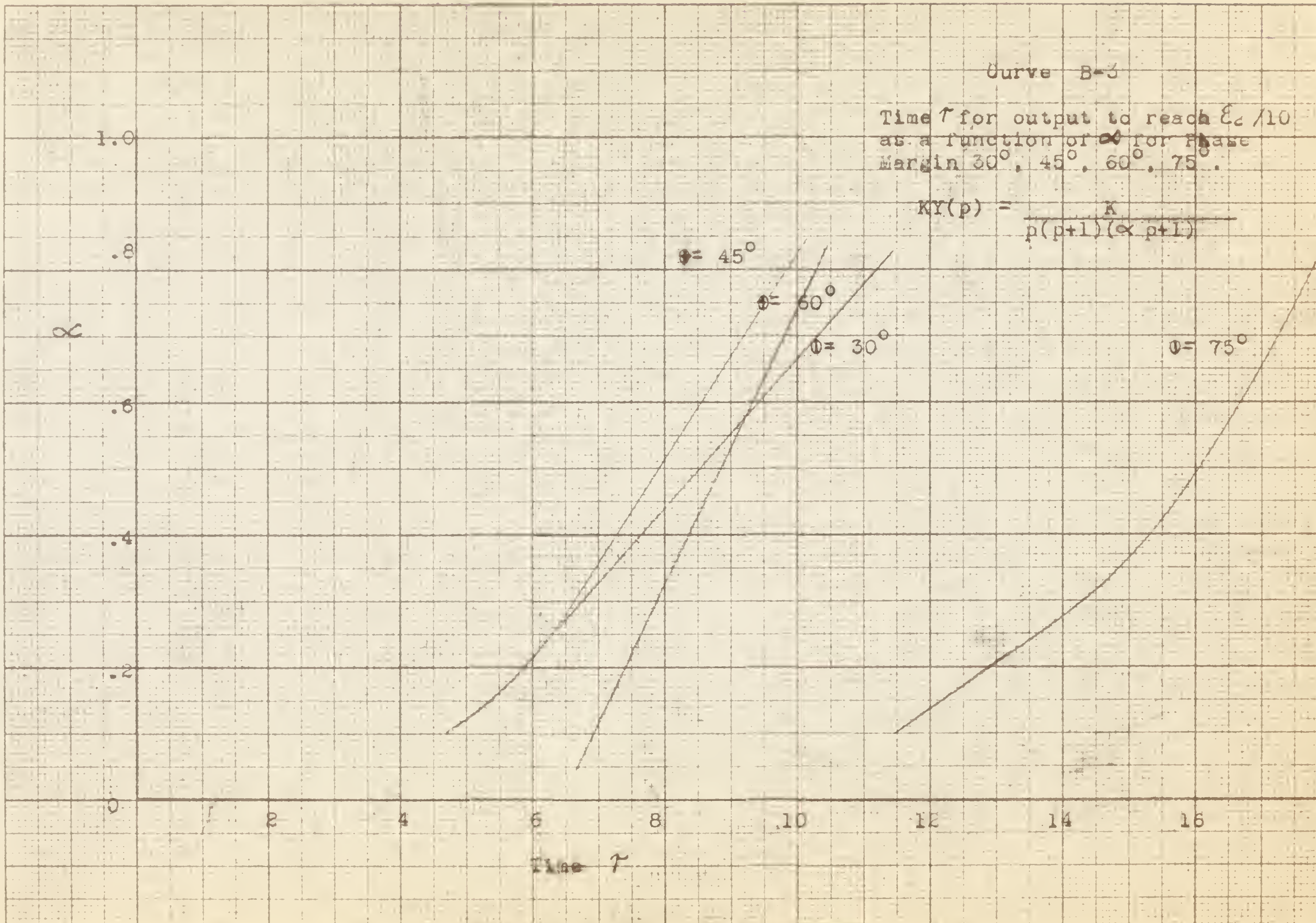

Curve B-1
Phase Margin and Gain Margin
as a function of α and K.
Phase Margin
Gain Margin
$KY(p) = \frac{K}{p(p+1)(\alpha p+1)}$
1.0
.8
α
.6
.4
.2
0
0.9
0.8
0.7
0.6
0.5
0.3
0.2
75°
60°
45°
30°
15°
1
2
3
4
5
6
7
8
9
Gain Constant K

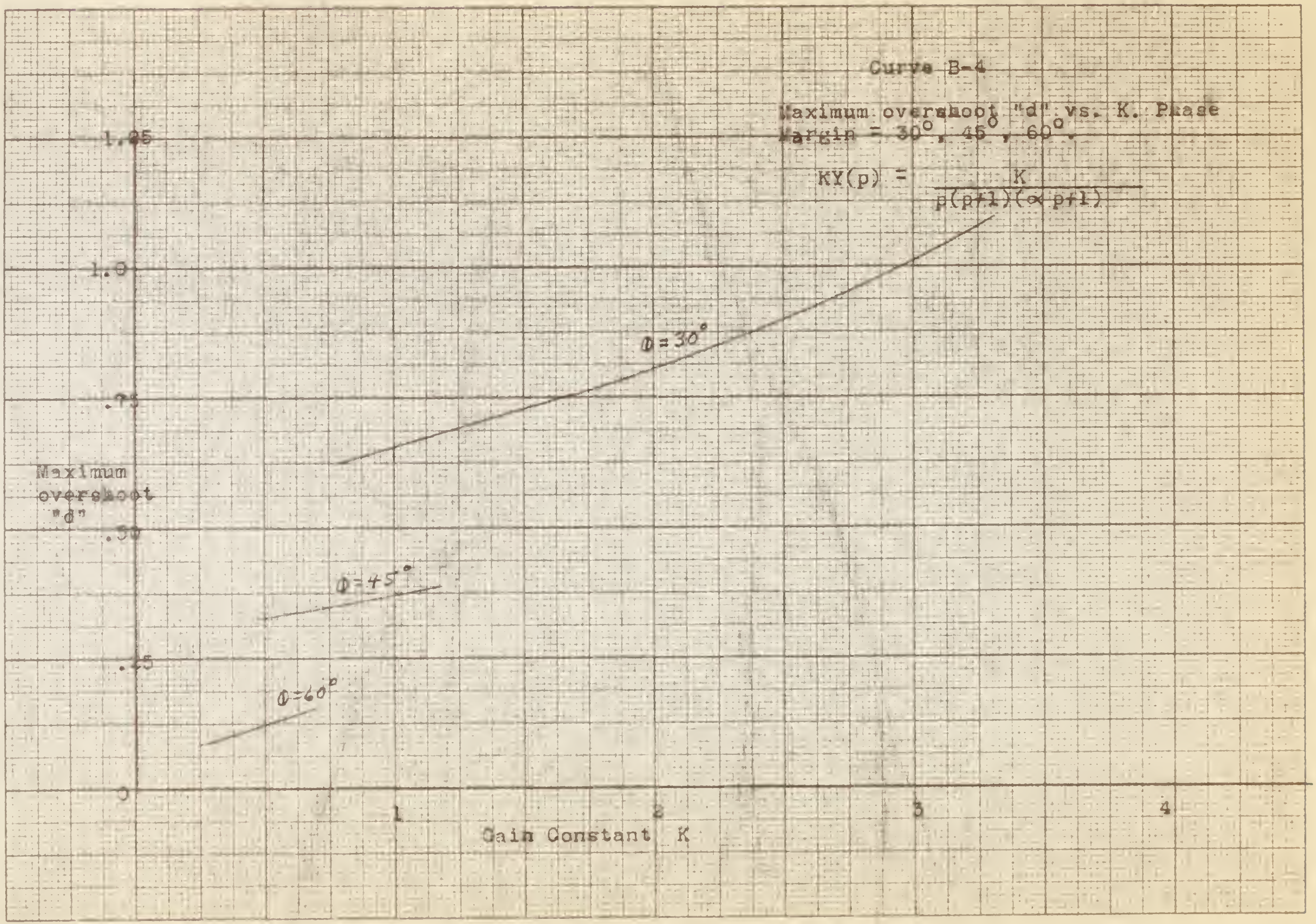


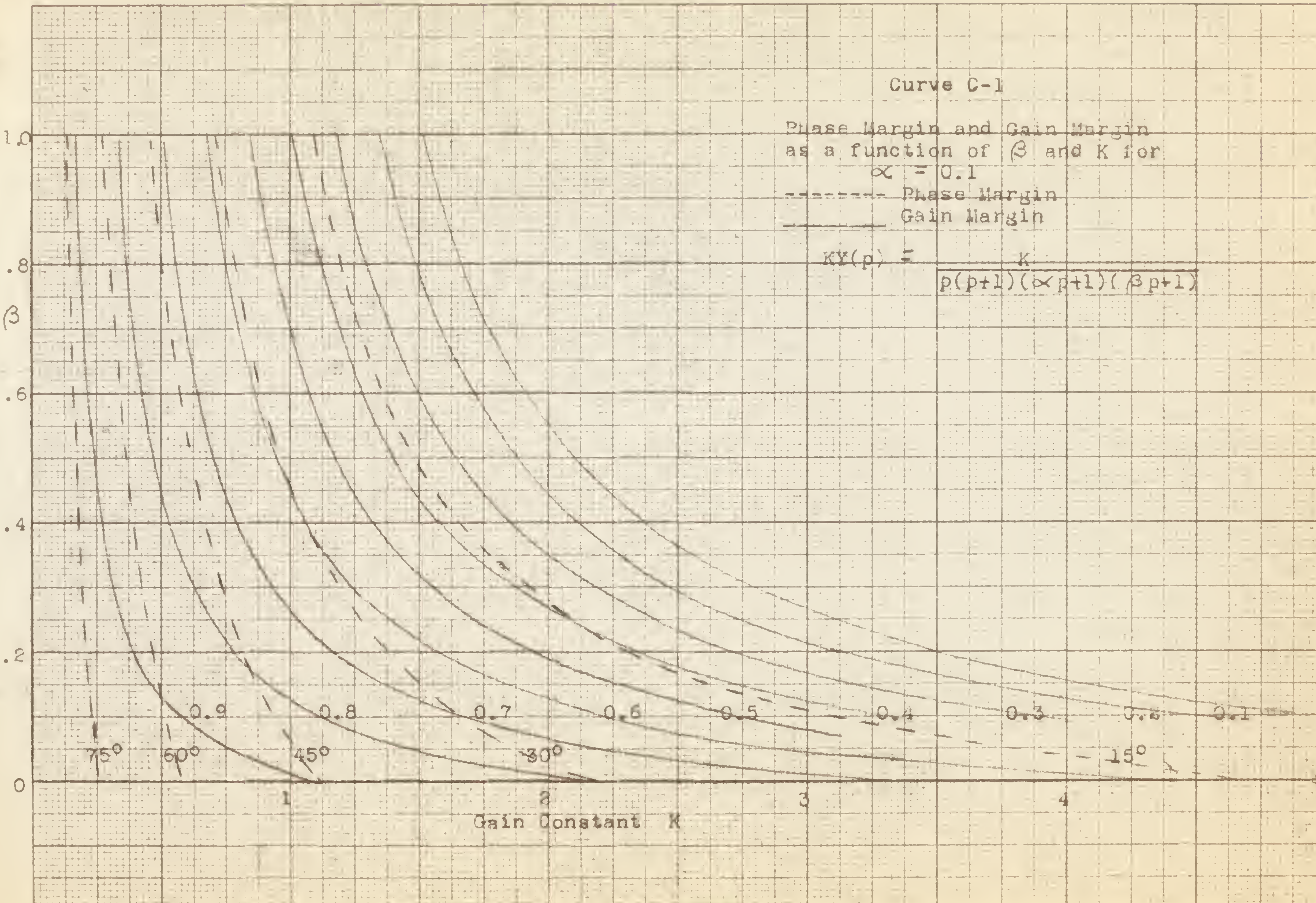
Curve B-2

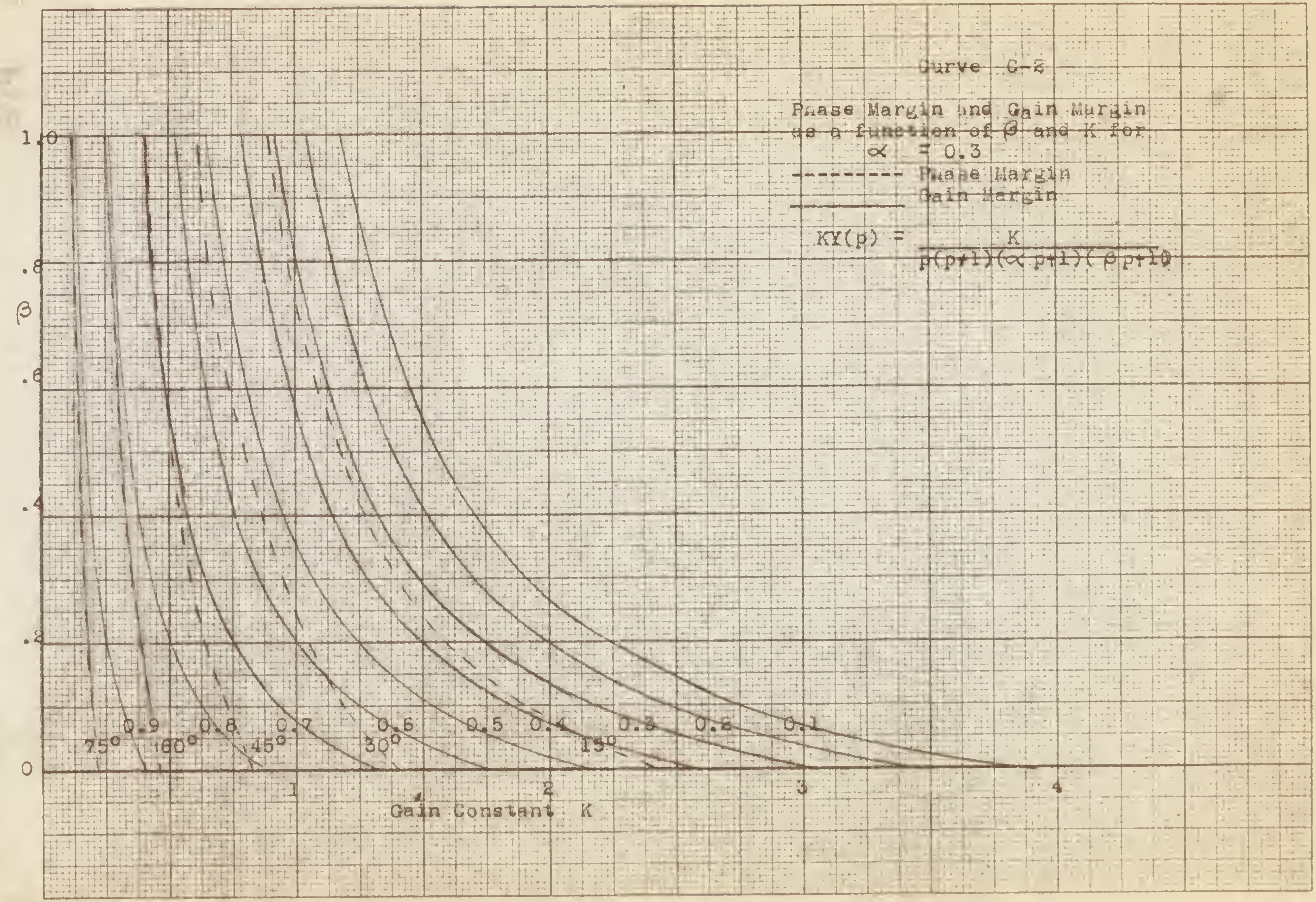
Maximum overshoot "d" as a function of α. Phase Margin = 30°, 45°, 60°.

$$KY(p) = \frac{K}{p(p+1)(\alpha p+1)}$$

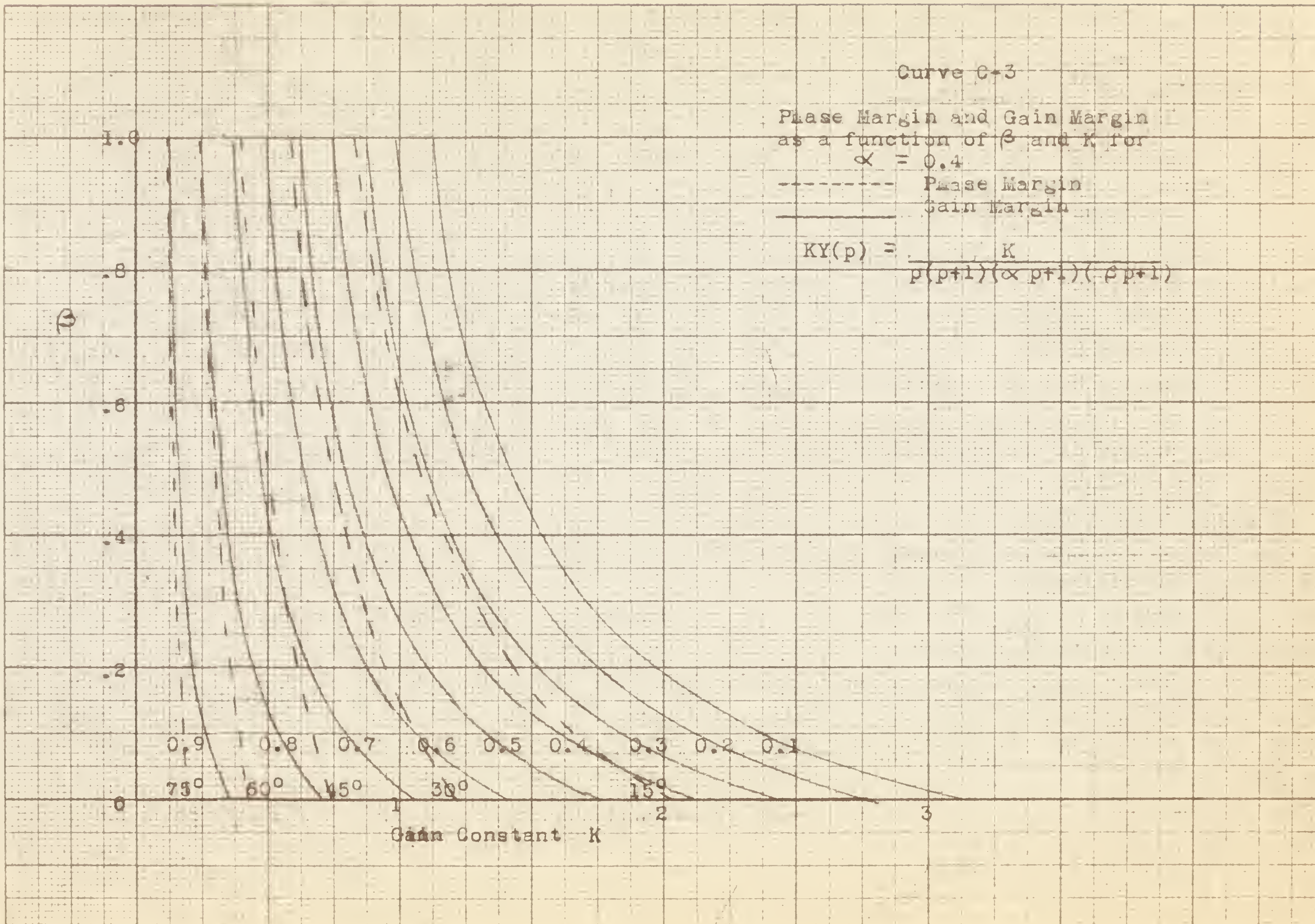

Curve B-3
Time τ for output to reach $\mathcal{E}_c$/10 as a function of α for Phase Margin 30°, 45°, 60°, 75°.
$KY(p) = \dfrac{K}{p(p+1)(\alpha p+1)}$
Φ= 45°
Φ= 60°
Φ= 30°
Φ= 75°
α
1.0
.8
.6
.4
.2
0
2
4
6
8
10
12
14
16
Time τ


Curve B-4
Maximum overshoot "d" vs. K. Phase Margin = 30°, 45°, 60°.
KY(p) = K / p(p+1)(αp+1)
Φ = 30°
Φ = 45°
Φ = 60°
Maximum overshoot "d"
1.25
1.0
.75
.50
.25
0
1
2
3
4
Gain Constant K


Curve C-1
Phase Margin and Gain Margin
as a function of β and K for
α = 0.1
-------- Phase Margin
________ Gain Margin
KY(p) = K / p(p+1)(αp+1)(βp+1)
1.0
.8
β
.6
.4
.2
0
1
2
3
4
Gain Constant K
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
75°
60°
45°
30°
15°



Curve C-2
Phase Margin and Gain Margin
as a function of β and K for
α = 0.3
---------- Phase Margin
———— Gain Margin
KY(p) = K / p(p+1)(αp+1)(βp+1)
1.0
.8
.6
.4
.2
0
β
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
75°
60°
45°
30°
15°
1
2
3
4
Gain Constant K


Curve C-3
Phase Margin and Gain Margin as a function of β and K for
α = 0.4
Phase Margin
Gain Margin
KY(p) = K / p(p+1)(α p+1)(β p+1)
β
1.0
.8
.6
.4
.2
0
1
2
3
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
75°
60°
45°
30°
15°
Gain Constant K

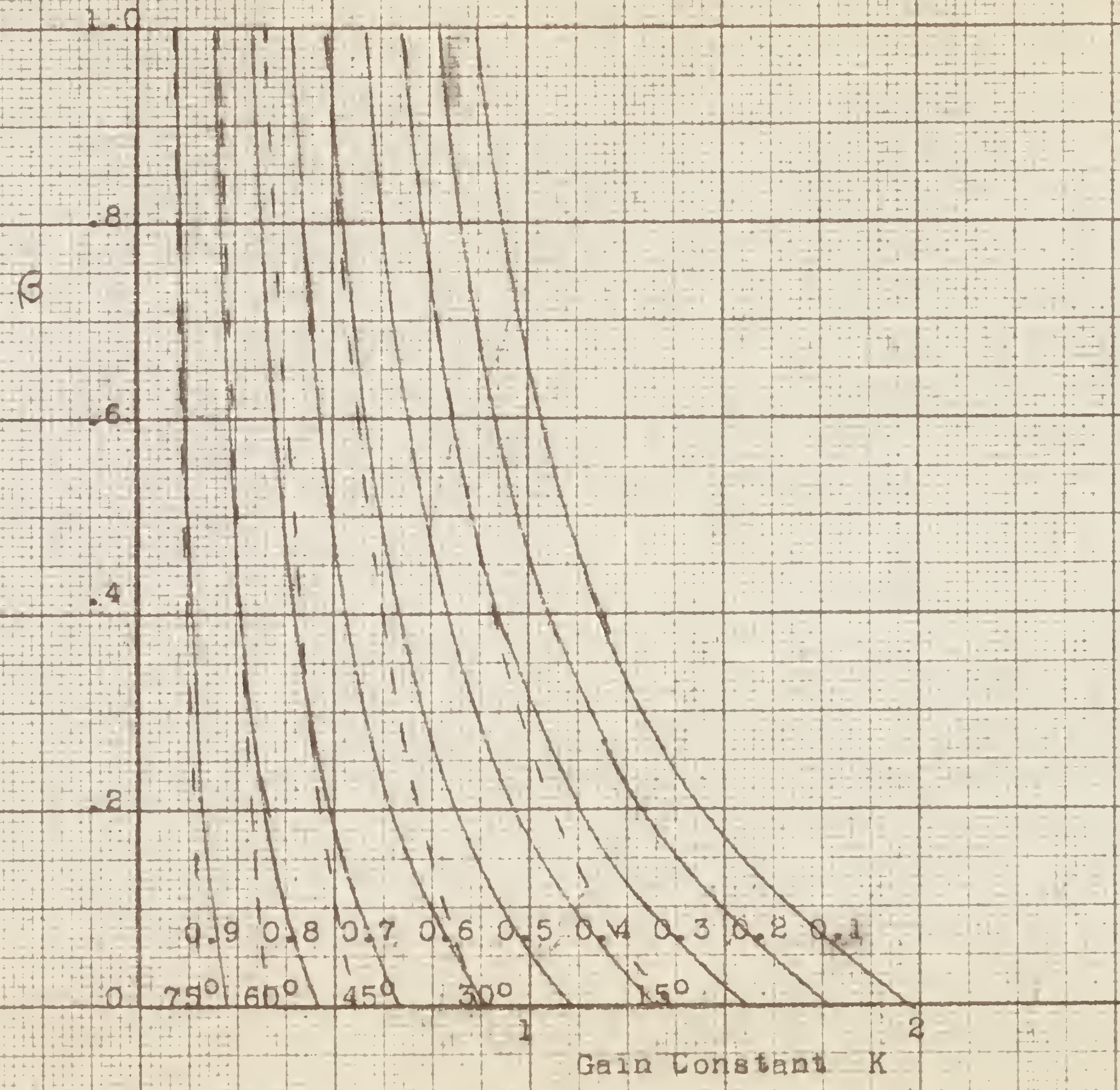
Curve C-4

Phase Margin and Gain Margin as a function of β and K for $\alpha = 0.8$

---------- Phase Margin

―――――― Gain Margin

$$KY(p) = \frac{K}{p(p+1)(\alpha p+1)(\beta p+1)}$$



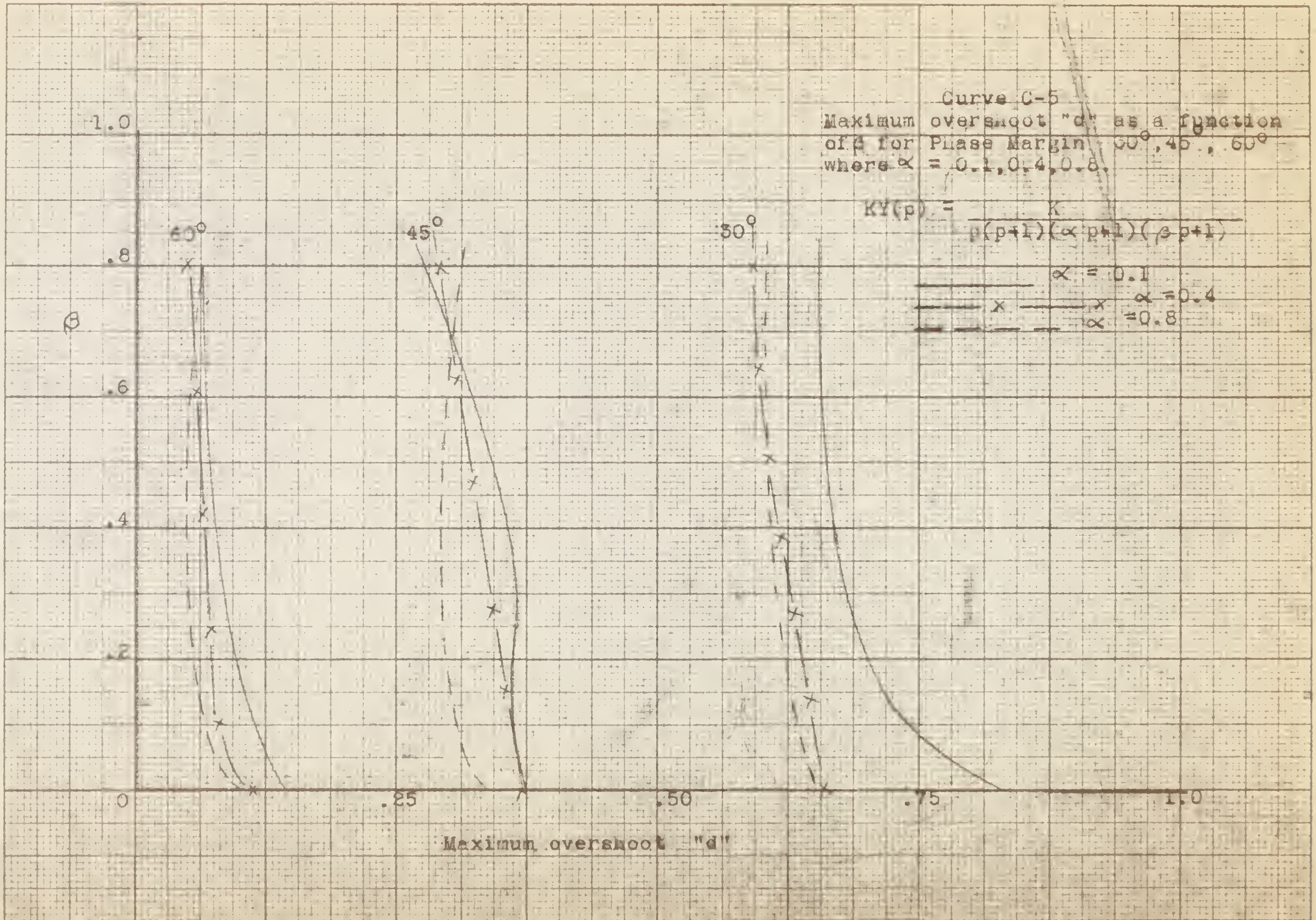
servo of expression (B). There is evidently an optimum value of phase margin when τ is considered. Here a phase margin of 45° gives better response characteristics than does 30° or 60°. The servo is overdamped for phase margin of 75°. From this curve it appears that τ is dependent upon α, but for phase margins of 30° - 60°, it is essentially independent of phase margin.

From Curve B-2 then, maximum overshoot d may by associated with phase margin while from Curve B-3 time τ for the servo to settle from its transient may be associated with α for values of phase margin of approximately 25° - 60°.

Curve B-4 shows maximum overshoot d as a function of K for values of phase margin $\Phi = 30^{\circ}, 45^{\circ}, 60^{\circ}$. As would be expected from the similarity of the phase margin and maximum overshoot d curves versus α and K, this curve is almost a straight line.

Curve C-5 shows maximum overshoot d for the various values of phase margin and β where α = 0.1, 0.4, 0.8 in the servo of expression (C). Here d is essentially independent of both α and β but a function of Φ. This tendency was also shown in Curve B-2 for expression (B). Here again d may be associated with phase margin.

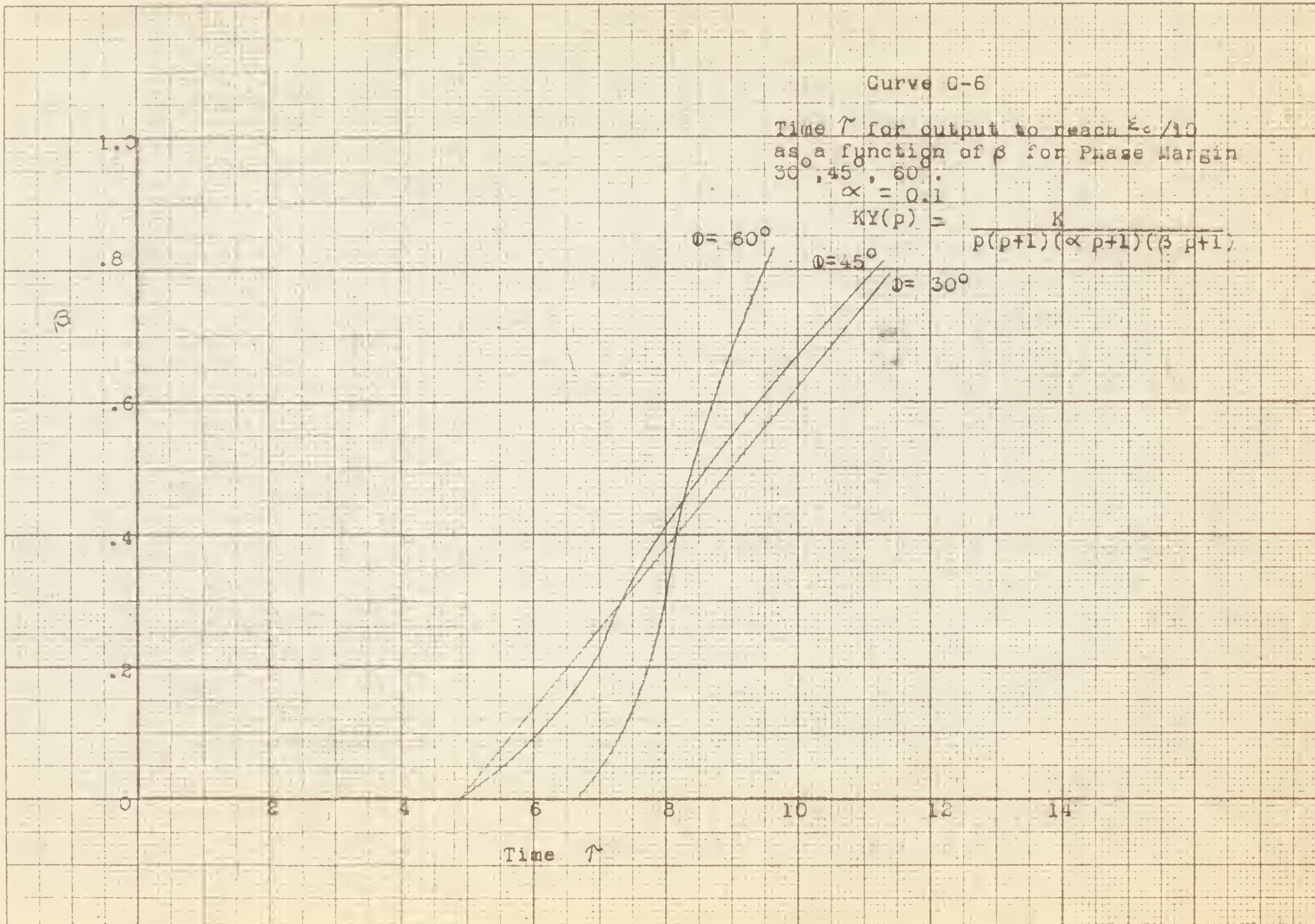
Curves C-6 through C-8 show τ as a function of β for α = 0.1, 0.4, 0.8 . τ is again essentially independent of phase margin though an optimum value of $\Phi = 60^{\circ}$ is indicated, for values of $\beta > 0.3$ τ increases with β for a given phase margin in a nearly linear fashion.



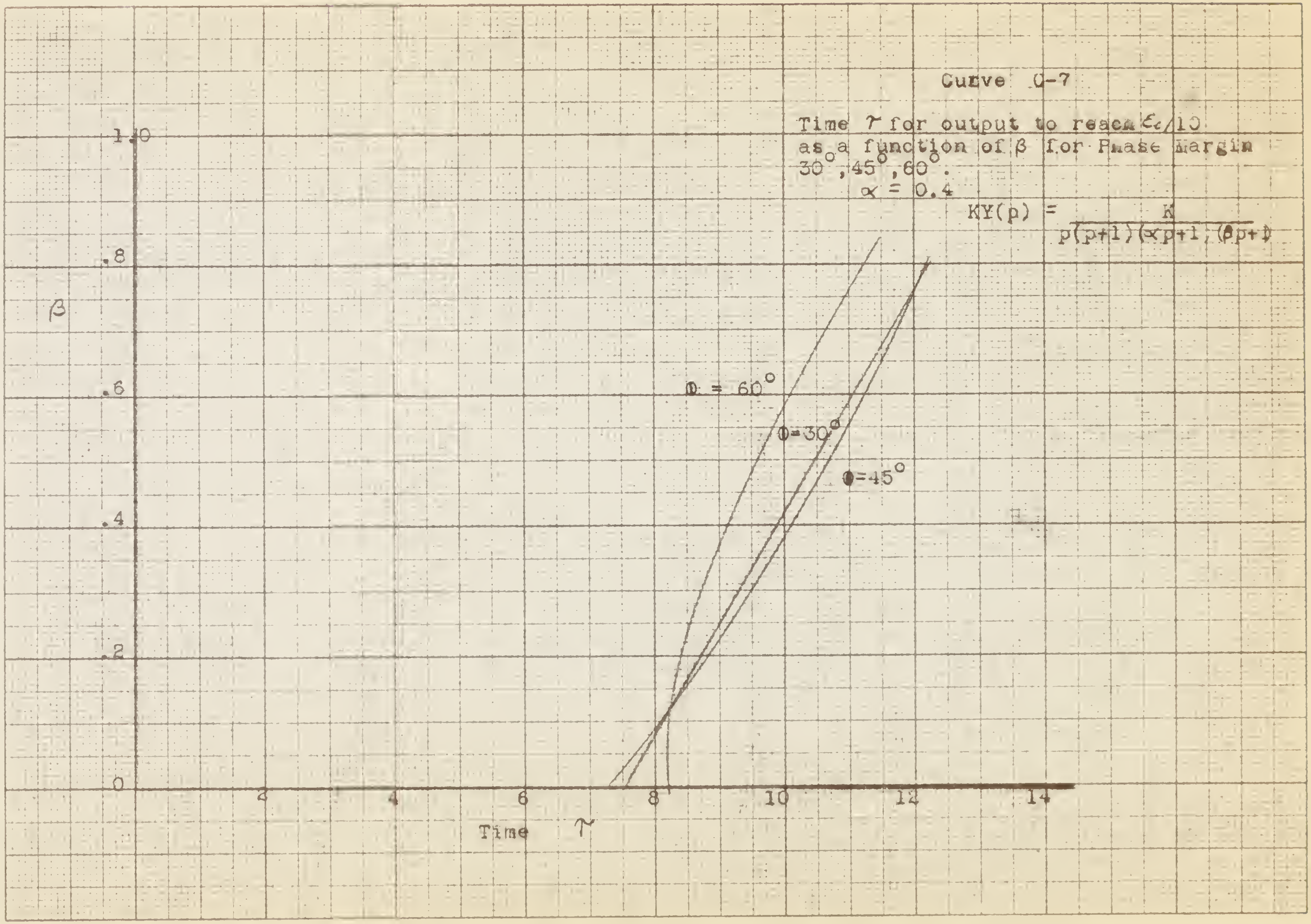
Curve C-5

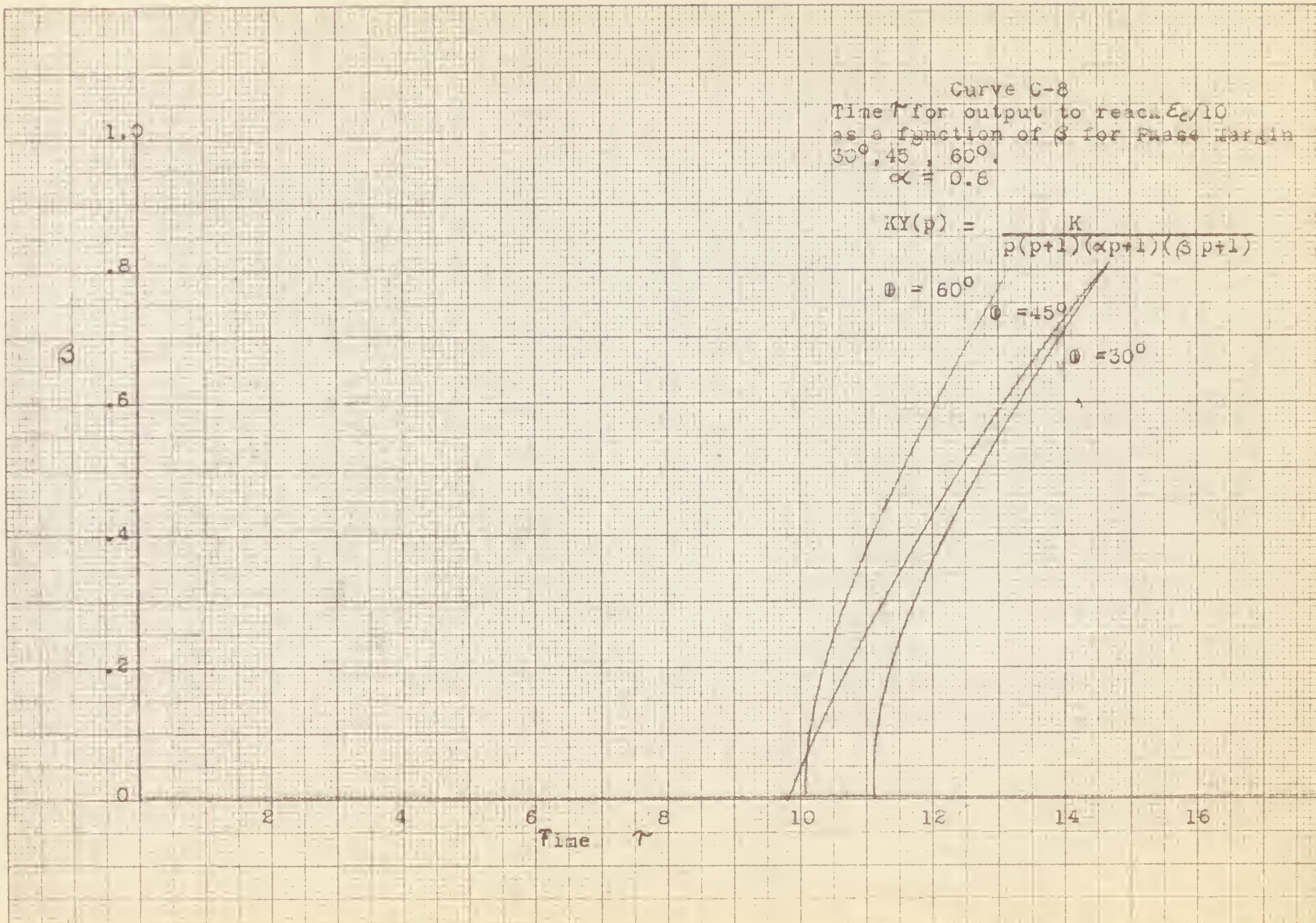
Maximum overshoot "d" as a function of β for Phase Margin 30°, 45°, 60° where α = 0.1, 0.4, 0.8.

$$KY(p) = \frac{K}{p(p+1)(\alpha p+1)(\beta p+1)}$$



Curve C-6
Time τ for output to reach $E_c/10$ as a function of β for Phase Margin $30^\circ, 45^\circ, 60^\circ$.
$\alpha = 0.1$
$KY(p) = \frac{K}{p(p+1)(\alpha p+1)(\beta p+1)}$
$\Phi = 60^\circ$
$\Phi = 45^\circ$
$\Phi = 30^\circ$
1.0
.8
.6
.4
.2
0
β
2
4
6
8
10
12
14
Time τ


Curve C-7
Time τ for output to reach $\mathcal{E}_c/10$ as a function of β for Phase Margin $30^\circ, 45^\circ, 60^\circ$.
$\alpha = 0.4$
$KY(p) = \frac{K}{p(p+1)(\alpha p+1)(\beta p+1)}$
$\Phi = 60^\circ$
$\Phi = 30^\circ$
$\Phi = 45^\circ$
β
1.0
.8
.6
.4
.2
0
2
4
6
8
10
12
14
Time τ


Curve C-8
Time τ for output to reach $\mathcal{E}_c/10$
as a function of β for Phase Margin
30°, 45°, 60°.
$\alpha = 0.8$
$KY(p) = \dfrac{K}{p(p+1)(\alpha p+1)(\beta p+1)}$
Φ = 60°
Φ = 45°
Φ = 30°
β
1.0
.8
.6
.4
.2
0
2
4
6
8
10
12
14
16
Time τ

In Curve C-9 τ is plotted vs. β for α = 0.1, 0.4, 0.8 . Phase margin was 45° in all cases. A dependence upon the values of both α and β is seen. This dependence was similarly noted in the servo having the transfer function of expression (B).

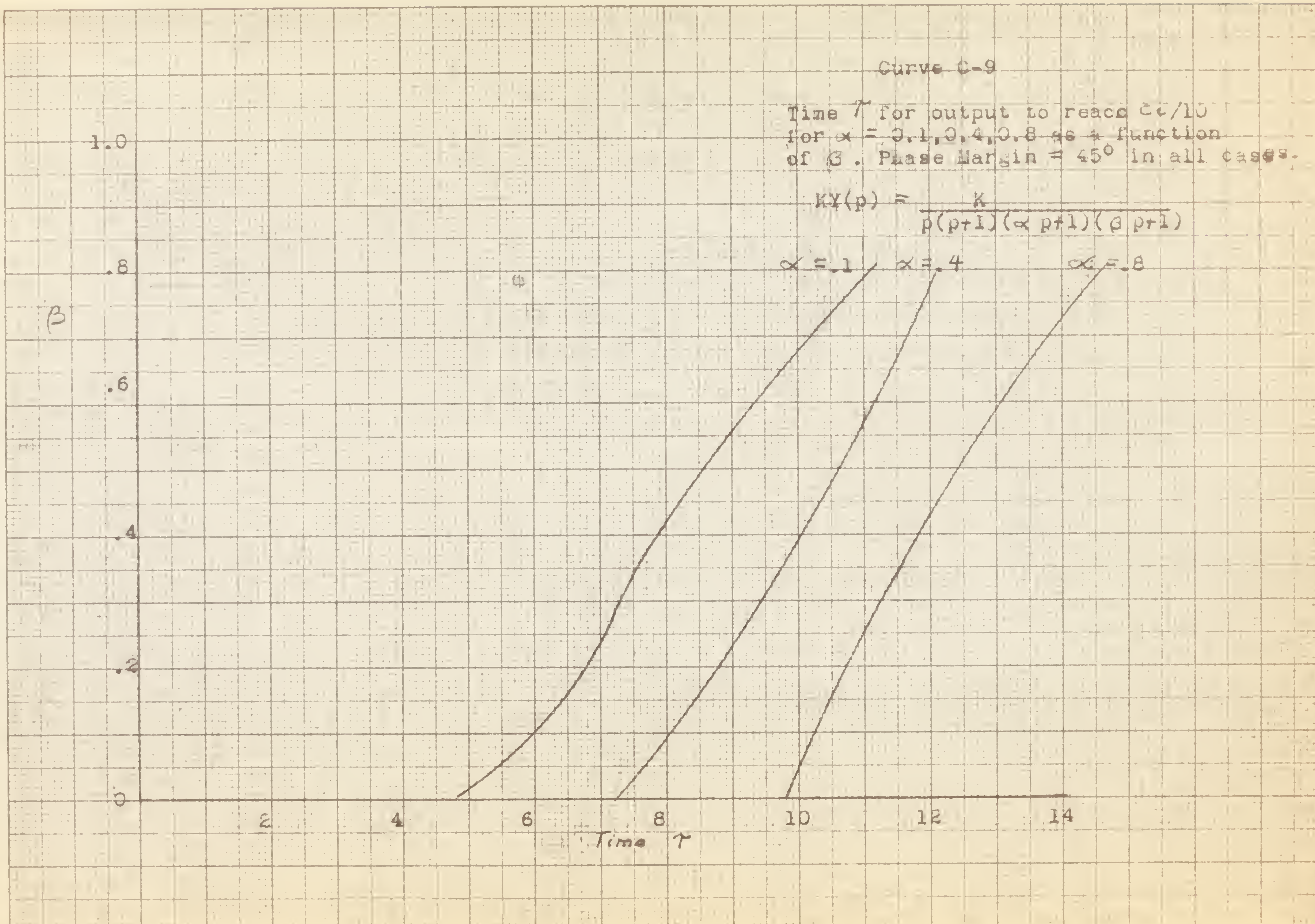
Curve D-1 shows transient response of the servos of expressions (B) and (C) for similar values of K. The runs paired together below had essentially the same transient response curves:

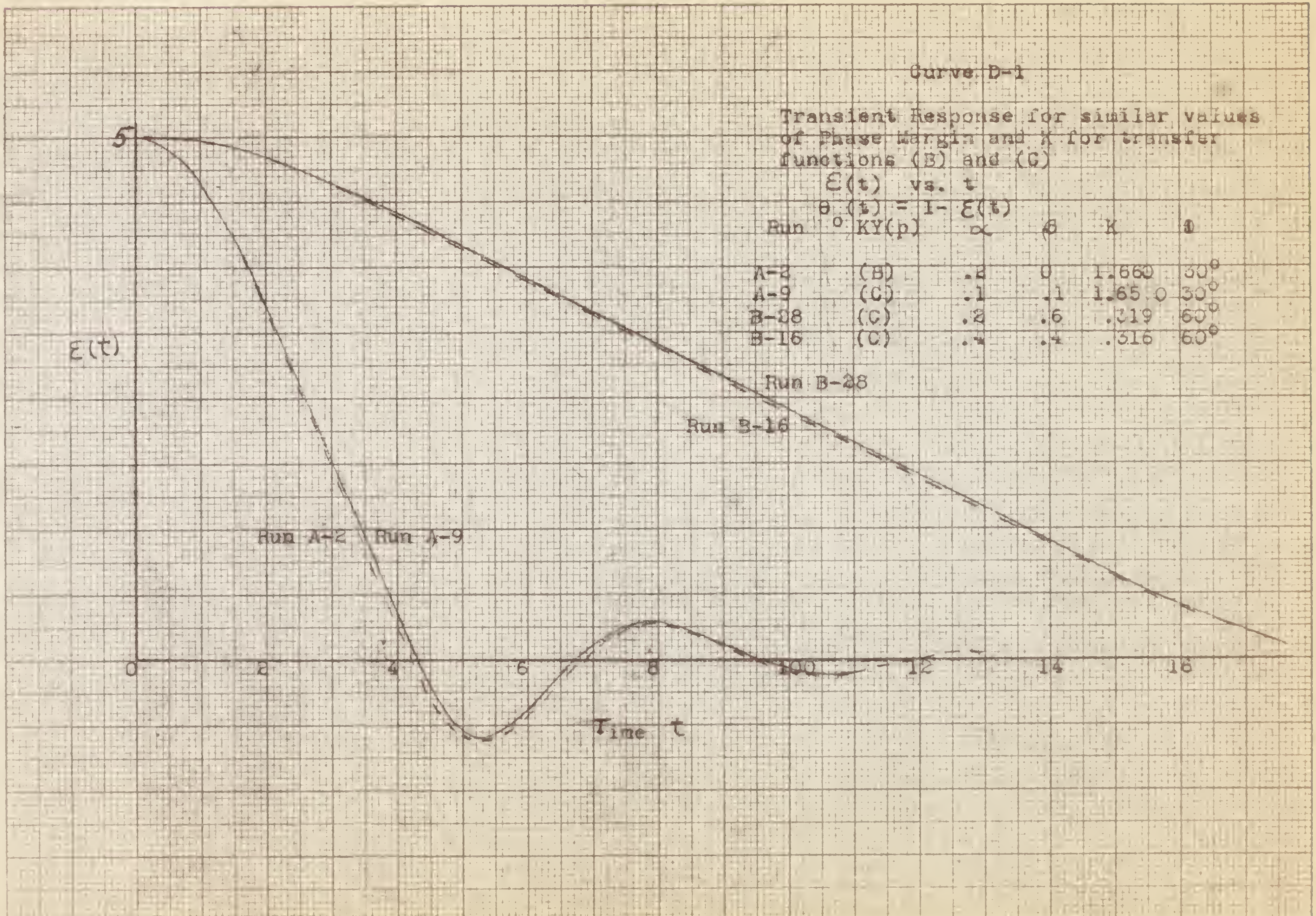
Run	KY(p)	Phase margin			K
A-2	(B)	30°	.2	0	1.66
A-9	(C)	30°	.1	.1	1.65
B-16	(C)	60°	.4	.4	.316
B-[illegible]8	(C)	60°	.2	.6	.319
B-2	(B)	45°	.4	0	.732
B-[illegible]1	(C)	45°	[illegible]	[illegible]	.719
B-3	(B)	60°	.4	0	[illegible]
B-2[illegible]	(C)	60°	.2	.2	.41[illegible]
B-9	(B)	30°	.3	.6	.7[illegible]1
B-53	(C)	30°	.1	.8	.78[illegible]

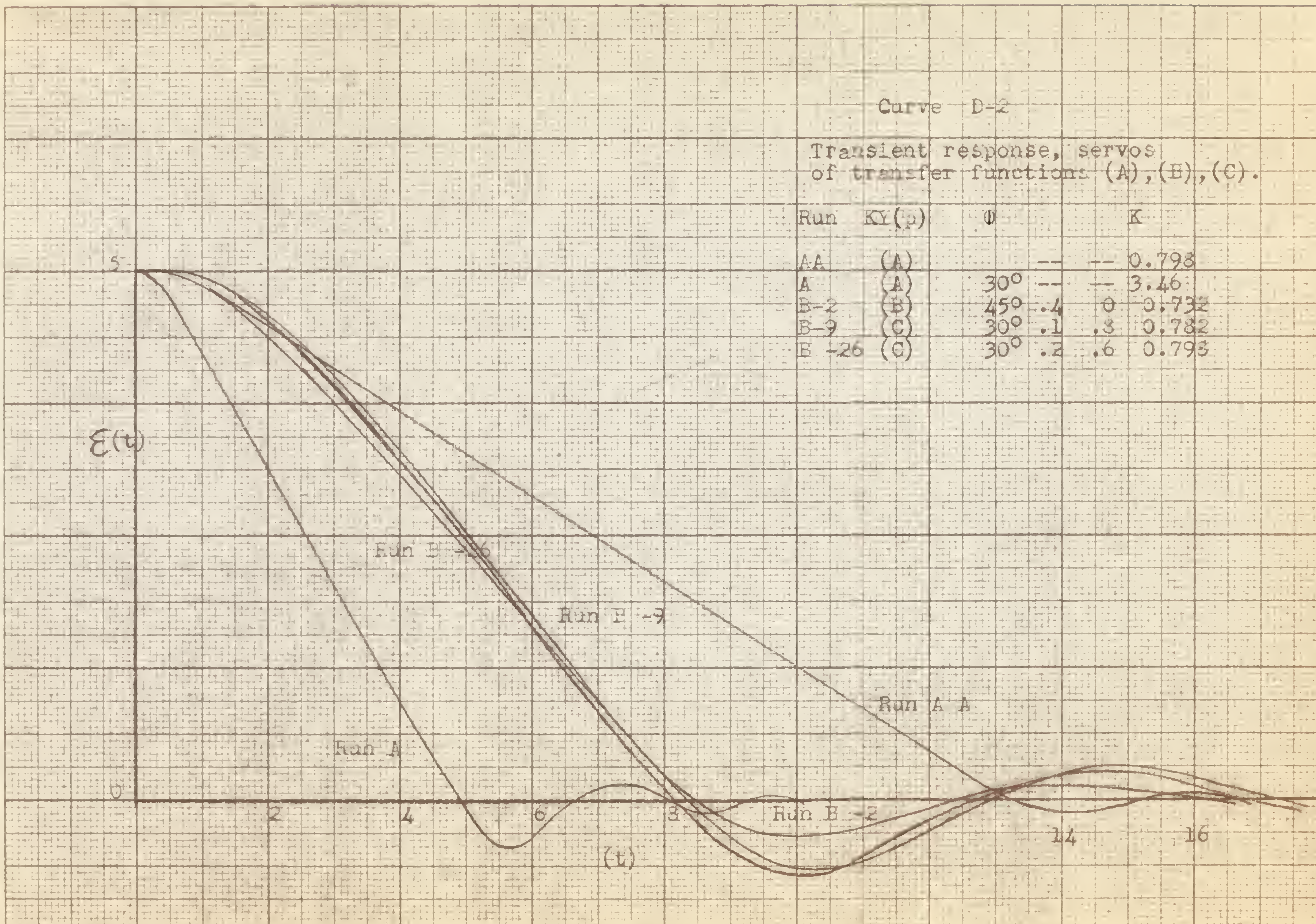
As further comparison runs B-2, B-9, and B-76 are compared in Curve D-2.

B-2	(B)	45°	.4	0	.732
B-9	(C)	30°	.1	.8	.78[illegible]
B-76	(C)	30°	.2	.5	.798

These runs, having similar K's but different α, β or phase margins have very similar transient responses to a step input. While these results show equal values of K to give approximately the same transient responses for the servos of expressions (B) and (C) this result does not hold for that having the transfer function of expression (A). Thus in curve D-[illegible], Run [illegible], K = 0.79[illegible], a different transient response results from that of Run B-76. Similarly, in Run A, use of a phase margin θ = 30° (K = [illegible].[illegible]6)


Curve C-9
Time τ for output to reach
for α = 0.1, 0.4, 0.8 as a function
of β. Phase Margin = 45° in all cases.
KY(p) = K / p(p+1)(α p+1)(β p+1)
α = .1
α = .4
α = .8
1.0
.8
.6
.4
.2
0
β
2
4
6
8
10
12
14
Time τ


Curve D-1
Transient Response for similar values of Phase Margin and K for transfer functions (B) and (C)
ε(t) vs. t
θo(t) = 1 - ε(t)
Run KY(p) α β K Φ
A-2 (B) .2 0 1.660 30°
A-9 (C) .1 .1 1.65 0 30°
B-28 (C) .2 .6 .319 60°
B-16 (C) .4 .4 .316 60°
Run B-28
Run B-16
Run A-2
Run A-9
5
ε(t)
0
2
4
6
8
10
12
14
16
Time t


Curve D-2
Transient response, servos
of transfer functions (A), (B), (C).
Run KY(p) Φ K
AA (A) -- -- 0.798
A (A) 30° -- -- 3.46
B-2 (B) 45° .4 0 0.732
B-9 (C) 30° .1 .3 0.782
B -26 (C) 30° .2 .6 0.793
5
0
ε(t)
2
4
6
8
14
16
(t)
Run A
Run B -26
Run B -9
Run A A
Run B -2

in the servo of expression (A) gives a response different from that obtained in Run B-9 where a phase margin of 30° was also used.

Discussion

From the previous curves of phase margins as a function of α and K it is seen that where phase margin is large (60^{o}-75^{o}) it is essentially linear with respect to K. For smaller values of phase margin this linearity no longer holds. For large values of α (or β) the curves of phase and gain margin are of similar shape. For small α (or β) this similarity ceases and for α (or β) in the vicinity $0 < \alpha$ or $\beta < 0.3$ there is marked dissimilarity. For transfer function (C) as α increases the curves of gain and phase margin become associated with smaller values of K.

The curves of maximum overshoot d as a function of α or β indicate that this property of the transient response may be associated with K. (It being understood however that K here is determined by α and β) The curves of time τ for the output to settle to a value $\varepsilon_c/10$ indicate that τ may be associated with α or β.

The associations of maximum overshoot d with K, and of τ with α and β may be made analytically by reference to Ch. 11, ref ([illegible]). There $h(t) = \frac{2}{\pi}\int_0^\infty \mathrm{Re}\,H(jw)\cos wt\,dw$ where

$$H(jw) = \frac{\theta_o(jw)}{\theta_i(jw)} = \frac{KY(jw)}{1+KY(jw)}$$

Using the loci plot of Fig. [illegible], Ch. 11 there may be obtained $\mathrm{Re}\,\frac{KY}{1+KY} = \mathrm{Re}\,\frac{\theta_o}{\theta_i}$ Here $\mathrm{Re}\,\frac{\theta_o}{\theta_i}$ is determined by K for a given α in expression ([illegible]) or a given α and β in expression (C). Thus K determines the ordinate in the trapezoidal method given in Ch. 11 for the determination of transient response and consequently determines d for a given servo system. Similarly, α and β determine the locations of the points $w_1, w_2, \ldots w_n$ in this method and consequently the time τ for the transient response to settle to $\varepsilon_c/10$.

The value of phase margin and of gain margin as design criteria is limited in that they each specify only one point on the transfer function locus. While useful in the discussion of a given servomechanism, their usefulness in synthesis of transfer functions must be augmented by more powerful methods, as for example the Lm and Ang vs log u, Lm-Ang, KY^{-1}, and constant M constant N methods of ref (a).

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Appendix (1)

NOMENCLATURE

θ_o - output, controlled quantity

θ_i - input, command quantity

ε - error, error signal

ε_e - value of error signal at which system saturates

ε_o - input error signal, $\theta_i - \theta_o$

K - gain constant

KY(p) - transfer function

(p) - operator $p = ju$ in LaPlace notation, $p = d/dt$ in differential notation.

T - time required for transient to settle to $\varepsilon_c/10$

d - maximum overshoot of transient response

u - dimensionless frequency, $u = wT_1$

t - unit of time

T_1 - largest time constant of transfer function

T_2 - second largest time constant of transfer function

α = constant, T_2/T_1

β = constant, T_3/T_1

Appendix (2)

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